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Scientific Enlightenment, Div. 2

A. 4: The Problem of Representation (and the Constitution of Classical Mechanics)

Chapter 9: The Relativity Theory of Albert Einstein

9.2. General relativity

2006 by L. C. C. Unfinished. To be continued.

General relativity is discovered when special relativity is discovered to be just another "cave" and the ascent is continued. As we ascend further up the Platonic cave, we notice again that what we have then taken as the real things of which what were earlier thought to be the real things were merely shadows, are again merely shadows of some other real things. This constitutes the essence of the story of the ascent from special relativity to general relativity. After we have realized that what have seemed to be the invariant space distances and time intervals are merely shadows -- images, likeness (*eikasia*) -- cast by the "real" Minkowski spacetime metric $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ which is the real invariant "interval" (under Lorentz transformation), we are now, with general relativity, coming to the realization that this Minkowski metric itself is only a shadow of the Gaussian spacetime metric. What does "shadow" mean here? It means that, while the Minkowski spacetime metric is valid for an Euclidean spacetime, the "real" spacetime -- spacetime on its total scale -- is non-Euclidean, within which the Minkowski metric has only a limited, local, "special" validity: it is valid for an infinitely small region of spacetime that may under all considerations be taken as Euclidean, just as, while the surface of the earth is non-Euclidean, not flat, but curved, an infinitely small region on it may well for all purposes be considered Euclidean, i.e. appear flat enough.

Der Beschleunigungszustand des unendlich kleinen ("örtlichen") Koordinatensystems ist hierbei so zu wählen, daß ein Gravitationsfeld nicht auftritt; dies ist für ein unendlich kleines Gebiet möglich. (1916a, p. 777)

The acceleration of the infinitely small ("local") coordinate system is hereby so chosen, that a gravitational field does not occur; this is possible for an infinitely small region.

Einstein is referring to the equivalence principle, the equivalence between acceleration and the effect of a gravitational field (see below), the absence of these two making spacetime Minkowskian, "empty", Euclidean, in which Euclidean inertial coordinate frames are in uniform translation movement relative to one another (special relativity).

Once the reference bodies or coordinate frames deviate from uniform translation movement relative to one another, once they start accelerate or decelerate relative to one another -- and when spacetime is considered in its large-scale they always do (below) -- the Minkowski metric breaks down, and the perspectives of the coordinate frames can no longer use Lorentz transformation to measure up a spacetime distance defined by the Minkowski metric and invariant under a change of "perspective" (coordinate frame), because spacetime under such condition appears warped or curved up. The invariant distance of spacetime -- now warped -- is instead to be calculated in each frame by the Gaussian metric. As Thibault Damour explains this Gaussian metric:

... le théorème de Pythagore-Einstein dans un espace-temps déformé, "maillé" par quatre coordonnées arbitraires x_0, x_1, x_2, x_3 , affirme que l'intervalle carré entre deux points infiniment voisins (de coordonnées x_0, x_1, x_2, x_3 , et $x_0 + dx_0, x_1 + dx_1, x_2 + dx_2, x_3 + dx_3$) est une somme de termes proportionnels aux carrés et aux doubles produits de différences (infinitésimales) de coordonnées: dx_0, dx_1, dx_2, dx_3 . Il y a *dix* termes dans cette somme car il y a quatre carrés, $dx_0^2, dx_1^2, dx_2^2, dx_3^2$ et six doubles produits $2dx_0dx_1, 2dx_0dx_2, 2dx_0dx_3, 2dx_1dx_2, 2dx_1dx_3, 2dx_2dx_3$. (TD, p. 214)

... the Pythagoras-Einstein theorem in a deformed space-time, tiled by four arbitrary coordinates x_0, x_1, x_2, x_3 , affirms that the squared interval between two points infinitely close together (the coordinates x_0, x_1, x_2, x_3 , and $x_0 + dx_0, x_1 + dx_1, x_2 + dx_2, x_3 + dx_3$) is the sum of terms proportional to the squares and the double products of the (infinitesimal) coordinate differences: dx_0, dx_1, dx_2, dx_3 . There are *ten* terms in this sum because there are four squares, $dx_0^2, dx_1^2, dx_2^2, dx_3^2$ and six double products $2dx_0dx_1, 2dx_0dx_2, 2dx_0dx_3, 2dx_1dx_2, 2dx_1dx_3, 2dx_2dx_3$.

The latter six coefficients, absent in special relativity, are produced by adding g_{21} with g_{12} ; g_{13} with g_{31} ; g_{14} with g_{41} ; g_{23} with g_{32} ; g_{24} with g_{42} ; g_{34} with g_{43} .

Les coefficients des quatre carrés sont dénotés, respectivement, $g_{00}, g_{11}, g_{22}, g_{33}$, alors que les coefficients des doubles produits sont notés $g_{01}, g_{02}, g_{03}, g_{12}, g_{13}, g_{23}$. Si on appelle ds^2 l'intervalle carré infinitésimal entre les deux points considérés, on peut écrire le théorème de Pythagore-Einstein sous la forme $ds^2 =$

$\Sigma g_{\mu\nu} dx_\mu dx_\nu$, où chaque indice, μ ou ν , peut prendre les quatre valeurs 0, 1, 2, 3 et où le signe Σ indique que l'on somme, indépendamment, sur les deux indices μ et ν . Einstein simplifia cette notation (due à Riemann) en remarquant qu'il était inutile d'écrire le symbole Σ car il suffisait d'admettre implicitement que l'on doit sommer sur tout indice répété (ici μ et ν). (TD, ibid.)

The coefficients of the four squares are denoted, respectively, g_{00} , g_{11} , g_{22} , g_{33} , while the coefficients of double products are named g_{01} , g_{02} , g_{03} , g_{12} , g_{13} , g_{23} . If one calls ds^2 the infinitesimal squared interval between two points being considered, one can write the Pythagoras-Einstein theorem in the form $ds^2 = \Sigma g_{\mu\nu} dx_\mu dx_\nu$, where each index, μ or ν , can take on the four values 0, 1, 2, 3 and where the sign Σ indicates that one sums, independently, over the two indices μ and ν . Einstein simplified this notation (due to Riemann) by remarking that it was useless to write the symbol Σ since it was sufficient to admit implicitly that one must sum over all repeated indices (here μ et ν).

$g_{11} (dx_1^2)$	$g_{12} (dx_1 dx_2)$	$g_{13} (dx_1 dx_3)$	$g_{14} (dx_1 dx_4)$
$g_{21} (dx_2 dx_1)$	$g_{22} (dx_2^2)$	$g_{23} (dx_2 dx_3)$	$g_{24} (dx_2 dx_4)$
$g_{31} (dx_3 dx_1)$	$g_{32} (dx_3 dx_2)$	$g_{33} (dx_3^2)$	$g_{34} (dx_3 dx_4)$
$g_{41} (dx_4 dx_1)$	$g_{42} (dx_4 dx_2)$	$g_{43} (dx_4 dx_3)$	$g_{44} (dx_4^2)$

- The "real" (Gaussian) spacetime metric is thus:¹
- $$ds^2 = g_{11} dx_1^2 + 2 g_{12} dx_1 dx_2 + 2 g_{13} dx_1 dx_3 + 2 g_{14} dx_1 dx_4 + g_{22} (dx_2)^2 + 2 g_{23} dx_2 dx_3 + 2 g_{24} dx_2 dx_4 + g_{33} (dx_3)^2 + 2 g_{34} dx_3 dx_4 + g_{44} (dx_4)^2$$

This reduces to the earlier Minkowski metric when in the matrix arrangement g_{11} , g_{22} , g_{33} , g_{44} have the value of 1 but the rest of 0 (i.e. the Kronecker delta δ_{ij} , with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if i is not equal to j):

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

That is, when spacetime is Euclidean, perfectly linear dimensions forming right

angles with each other, the totality exhibiting no curvature. *A special case.* "The Gaussian coordinate system is a logical generalisation of the Cartesian [= Euclidean] co-ordinate system." ("Das Gaußsche Koordinatensystem ist eine logische Verallgemeinerung des Kartesischen Koordinatensystems"; RSGT, p. 61; p. 100.) Again, think about the meaning of the relationship between the special and general relativity:

Es ist auch auf Nicht-Euklidische Kontinua anwendbar, allerdings nur dann, wenn kleine Teile des betrachteten Kontinuums mit Bezug auf das definierte Maß ("Abstand") sich mit desto größerer Annäherung Euklidisch verhalten, je kleiner der ins Auge gefaßte Teil des Kontinuums ist. (Ibid.)

It is also applicable to non-Euclidean continua, but only when, with respect to the defined "size" ("distance"), small parts of the continuum under consideration behave with ever greater approximation like a Euclidean system, the smaller the part of the continuum under our notice.

This is just as the calculus of the area under the curve (Σ : integration) is a generalization of the calculus of the area under a straight line: the smaller the part of the curve we focus on, the more it resembles a straight line, and the more the area under it approximates to the form of that under a straight line. On the other hand, the straight line may, as one steps back ever further to get the larger picture, appear ever to be just a small part of *a curve*, not a straight line: a special case. So the case of special relativity's becoming general is also of the same kind of ascent as Fermat's discovery that the law of inertia is a restricted case of Snell's law of refraction, and Kepler's discovery that the circle and a circular planetary orbiting at a constant velocity are only a special case of the ellipses and the elliptical planetary orbiting at varying velocity ("equal area in equal time"), as we have already explained. The Platonic ascent involved in the discovery of general relativity here is thus of the type that the expansion of its experiential horizon allows consciousness to notice that what it formerly considers to be the *whole* truth is really only a partial manifestation of the whole truth.

The curved spacetime of general relativity manifests itself partially to us as the Euclidean-Cartesian spacetime of Minkowski, which in turn manifests itself partially as the Euclidean-Cartesian (invariant) space *and* (invariant) time of everyday experience. The curvature of spacetime constitutes -- *is* -- the gravitational field, which is then the generalized case of the special case:

Die Verallgemeinerung kann man wie folgt charakterisieren. Das reine Gravitationsfeld der g_{ik} hat gemäß seiner Herleitung aus dem leeren "Minkowski-Raum" die Symmetrie-Eigenschaft $g_{ik} = g_{ki}$ ($g_{12} = g_{21}$ usw.). Das verallgemeinerte Feld ist von derselben Art, aber ohne die genannte Symmetrie-Eigenschaft. (RSGT, p. 101)

One can characterize the generalization in the following way. The pure gravitational field of g_{ik} has, in accordance with its derivation from empty "Minkowski space", the symmetry property $g_{ik} = g_{ki}$ ($g_{12} = g_{21}$, etc.). The generalized field is of the same kind, but without this symmetry property.

But, remember, the Minkowski spacetime metric corresponds to the Kronecker delta (which marks the metric structure of +++) only because of the use of the imaginary number in the time element, such that the positive $d(\sqrt{-1}(ct))^2$ takes up the place of $-c^2 dt^2$. If the negative sign of the time differential were kept, and the metric structure became +-+ as it is in its original form, the metric tensor for the Minkowski metric would then be:

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{vmatrix}$$

Again, Minkowski's four dimensional world is simply not the same thing as a four dimensional Euclidean space. But the question can now be answered of why the curvature of spacetime constitutes a gravitational field. As Einstein summarizes:

Aus dem [vorgehendende] Betrachtungen... geht hervor, daß die Größen $g_{\sigma\tau}$ [oder g_{ik}] vom physikalischen Standpunkte aus als diejenigen Größen anzusehen sind, welche das Gravitationsfeld in bezug auf das gewählte Bezugssystem beschreiben. Nehmen wir nämlich zunächst an, es sei für ein gewisses betrachtetes vierdimensionales Gebiet bei geeigneter Wahl der Koordinaten die spezielle Relativitätstheorie gültig. Die $g_{\sigma\tau}$ haben dann die in... angegebenen [konstante] Werte. Ein freier materieller Punkt bewegt sich dann bezüglich dieses Systems geradlinig gleichförmig. Führt man nun durch eine beliebige Substitutionen neue Raum-Zeitkoordinaten x_1, \dots, x_4 ein, so werden in diesem neuen System $g_{\mu\nu}$ nicht mehr Konstante, sondern Raum-Zeitfunktionen sein. Gleichzeitig wird sich die Bewegung des freien Massenpunktes in den neuen Koordinaten als eine krummlinige, nicht gleichförmige, darstellen, wobei dies Bewegungsgesetz unabhängig sein wird von der Natur des bewegten Massenpunktes. Wir werden also diese Bewegung als eine solche unter dem Einfluß eines Gravitationsfeldes deuten. Wir sehen das Auftreten eines Gravitationsfeldes geknüpft an eine raumzeitliche Veränderlichkeit der $g_{\sigma\tau}$. Auch in dem allgemeinen Falle, daß wir nicht in einem endlichen Gebiete bei passender

Koordinatenwahl die Gültigkeit der speziellen Relativitätstheorie herbeiführen können, werden wir an der Auffassung festzuhalten haben, daß die $g_{\sigma\tau}$ das Gravitationsfeld beschreiben.

Die Gravitation spielt also gemäß der allgemeinen Relativitätstheorie eine Ausnahmerolle gegenüber den übrigen, insbesondere den elektromagnetischen Kräften, indem die das Gravitationsfeld darstellenden 10 Funktionen $g_{\sigma\tau}$ zugleich die metrischen Eigenschaften des vierdimensionalen Meßraumes bestimmen. (1916a, p. 779)

From the [foregoing] considerations... it follows that the quantities $g_{\sigma\tau}$ [or $g_{\nu\kappa}$] are from the physical standpoint to be regarded as those which describe the gravitational field in relation to the chosen reference system. Namely, if we assume that special relativity be valid for a certain four dimensional region being considered with the properly chosen coordinates, then $g_{\sigma\tau}$ have the [constant] values given above. A free material point then moves in relation to this system uniformly and in a straight line. If one now introduces new spacetime coordinates $x_1 \dots x_4$ by means of any substitution [*i.e. if one switches to a different reference or coordinate frame to look at and measure the motion of the material point*], then $g_{\mu\nu}$ in this new system will no longer be constants [*i.e. acceleration now occurs*], but functions of spacetime. At the same time the motion of the free mass point will in the new coordinates present itself as curvilinear, not uniform, where this law of motion will be independent of the nature of the moving mass point. We will thus interpret this motion as one under the influence of a gravitational field. We thus see the occurrence of a gravitational field connected with a spacetime variability of $g_{\sigma\tau}$. Also in the general case, where we cannot by a suitable choice of coordinates maintain the validity of the special relativity theory in a finite region, we shall have to hold fast to the view that $g_{\sigma\tau}$ describe the gravitational field.

The gravitational field thus plays, according to the general relativity theory, an exceptional role with regard to other forces, especially the electromagnetic forces, in that the 10 functions of $g_{\sigma\tau}$ representing the gravitational field at the same time determine the metric properties of the four dimensional space measured.

Acceleration, gravitational force, and spacetime curvature are thus all identical phenomenon. These, in fact, necessarily exist, so that, as Einstein explains, the Minkowskian "special case" is always only an approximation, not a case that exists under some special circumstances, though not always.

Wenn man das Gravitationsfeld d. h. die Funktionen g_{ik} weggenommen denkt, so bleibt nicht etwa ein Raum vom Typus [Minkowskis], sondern überhaupt *nichts* übrig, auch kein "topologischer Raum". Denn die Funktionen g_{ik} beschreiben nicht nur das Feld, sondern gleichzeitig auch die topologische und metrische Struktur -- Eigenschaften der Mannigfaltigkeit. Ein Raum vom Typus [Minkowskis] ist im Sinne der allgemeinen Relativitätstheorie nicht etwa ein Raum ohne Feld, sondern ein Spezialfall des g_{ik} -Feldes, für welchen die $g_{ik} \dots$ Werte haben, die nicht von den Koordinaten abhängen; einen leeren Raum, d. h. einen Raum ohne Feld, gibt es nicht. (RSGT, p. 100)

If one imagines the gravitational field, i.e. the functions g_{ik} to be removed, there does not remain a space of the [Minkowski] type, but absolutely *nothing*, and also no "topological space". For the functions g_{ik} describe not only the field, but at the same time also the topological and metric structure -- properties of the manifold. A space of the [Minkowski] type is in the sense of general relativity theory not a space without field, but a special case of the g_{ik} -field, for which the $g_{ik} \dots$ have values that do not depend on the coordinates. There is no such thing as an empty space, i.e. a space without field. (RSGT eng., p. 176)

Finally, with general relativity the principle of relativity itself gets "generalized", such that the principle of relativity as found in special relativity turns out also to be a special case of the "general principle of relativity":

Nach der speziellen Relativitätstheorie gehen die die allgemeinen Naturgesetze ausdrückenden Gleichungen in Gleichungen derselben Form über, wenn man statt der Raum-Zeit-Variablen x, y, z, t eines (Galileischen) Bezugskörpers K unter Benutzung der Lorentz-Transformation die Raum-Zeit-Variablen x', y', z', t' eines neuen Bezugskörpers K' einführt. Nach der allgemeinen Relativitätstheorie dagegen müssen die Gleichungen bei Anwendung beliebiger Substitutionen der Gaußschen Variablen x_1, x_2, x_3, x_4 in Gleichungen derselben Form übergehen; denn jede Transformation (nicht nur die Lorentz-Transformation) entspricht dem Übergang eines Gaußschen Koordinatensystems in ein anderes. (RSGT, p. 66)

According to the special theory of relativity, the equations which express the general laws of nature pass over into equations of the same form when, by making use of the Lorentz transformation, one introduces, in place of the space-time variables x, y, z, t of a (Galilean) reference-body K , the space-time variables x', y', z', t' of a new reference-body K' . According to the general theory of

relativity, on the other hand, during application of arbitrary substitutions of the Gaussian variables x_1, x_2, x_3, x_4 , the equations must pass over into equations of the same form; for every transformation (not only the Lorentz transformation) corresponds to the transition from one Gaussian co-ordinate system into another.

In other words, whereas in special relativity with its Lorentz transformation and the constancy of speed of light in vacuum the principle of relativity -- the invariance of the form of the laws of nature -- is valid only for inertial frames in uniform translation movement relative to one another, in general relativity with its Gaussian metric the principle of relativity is now valid -- the form of the laws of nature is now invariant -- during changes between coordinate frames in any sort of movement whatever relative to one another, whether when one is accelerating relative to another or when one is in uniform rotating movement relative to another. The postulate or principle of general relativity, in a word:

Die Gesetze der Physik müssen so beschaffen sein, daß sie in bezug auf beliebig bewegte Bezugssystem gelten. (1916a, p. 772)

The laws of physics must be so constructed, that they are valid in relation to reference systems in whatever movement.

Uniform translation movement is a special, restricted case of *any movement whatsoever*.

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Footnotes:

1. That is, the generalized Pythagorean theorem defining the squared differential distance (ds^2) along a path on the spacetime manifold, when expressed in tensor form and with the (more concrete) Cartesian coordinate differentials of classical mechanics dt, dx, dy, dz , is: $ds^2 =$

$g_{11} (dt)(dt)$	$+ g_{12} (dt)(dx)$	$+ g_{13} (dt)(dy)$	$+ g_{14} (dt)(dz)$
$+ g_{21} (dx)(dt)$	$+ g_{22} (dx)(dx)$	$+ g_{23} (dx)(dy)$	$+ g_{24} (dx)(dz)$
$+ g_{31} (dy)(dt)$	$+ g_{32} (dy)(dx)$	$+ g_{33} (dy)(dy)$	$+ g_{34} (dy)(dz)$
$+ g_{41} (dz)(dt)$	$+ g_{42} (dz)(dx)$	$+ g_{43} (dz)(dy)$	$+ g_{44} (dz)(dz)$

That is (in Cartesian coordinates): $ds^2 = g_{tt} (cdt)^2 + 2 g_{tx} dx dt + 2 g_{ty} dy dt + 2 g_{tz} dz dt + g_{xx} (dx)^2 + 2 g_{xy} dx dy + 2 g_{xz} dx dz + g_{yy} (dy)^2 + 2 g_{yz} dy dz + g_{zz} (dz)^2$, which reduces to the Minkowski-Pythagorean metric $ds^2 = (cdt)^2 - [(dx)^2 + (dy)^2 + (dz)^2]$ when $g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$ and all the other $g_{\mu\nu}$ coefficients = 0. (KB)

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